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On Intuitionistic Fuzzy Relations

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Abstract

In this paper, we study Intuitionistic Fuzzy Relations defined on an intuitionistic fuzzy set which is observed to be a generalization of the intuitionistic fuzzy relation already in existence and are also extensions of generalized fuzzy relations. Some properties of such relations are also studied.

Key words: Intuitionistic fuzzy set, intuitionistic fuzzy relation, composition

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INTRODUCTION

Fuzzy binary relations explain the extent to which elements in the set are related. It was later generalized in Chakraborthy *et al*., (1983) by considering it as a relation between fuzzy sets. K. T. Atanassov (1986) later introduced Intuitionistic fuzzy sets (IFS) by incorporating non membership grade in a fuzzy set. Intuitionistic fuzzy relations (IFRs) has already been studied by many researchers. Commonly IFRs are IFSs in a Cartesian product of universes (Bustince *et al*.,1986). Here an attempt is made to extend IFRs to a relation between two IFSs.

The notion of generalized IFRs is introduced in Section 2. Then various binary and unary operations of these relations are defined. Throughout this paper, unless otherwise stated, by a relation, we mean intuitionistic fuzzy binary relation defined on IFSs over the universe *U*.

Definition 1.1. (Atanasov,1986) Let X be an ordinary (non fuzzy) set An intuitionistic fuzzy set A in X is given by

A = $\left\{(x, \mu_A(x), \nu_A(x)) / x \in X\right\}$

where $\mu_A : X \to [0,1],\ \nu_A : X \to [0,1]$

with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all x∑X.

Definition 1.2. (Bustince *et al*.,1986) An intuitionistic fuzzy relation is an intuitionistic fuzzy subset of *X* × *Y* , that is, is an expression *R* given by

$R = \{((x,y), \mu_R(x,y), \nu_R(x,y)) \mid x \in X, y \in Y\}$
where $\mu_R : X \times Y \to [0,1]$, $\nu_R : X \times Y \to [0,1]$
satisfy the condition $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$
for any $(x,y) \in X \times Y$

Definition 1.3. (Chakraborthy *et al*., 1983) Let *U* be the initial set and A, B be fuzzy subsets of *U* defined by the membership functions μ_A and μ_B respectively where the membership set is
[0,1]. A×B is the fuzzy subset of U × U defined by
$\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$ for all *x,y* ∈ U.
Let *R* ⊆ A×B.
That is, $\mu_R(x,y) \le \min\{\mu_A(x), \mu_B(y)\}$. Then *R* is a fuzzy relation from A to B.

Relations on intuitioinistic fuzzy sets.

Let U be any nonempty set and A, B be IFS in U given by the membership functions μ_A, μ_B respectively and the nonmembership functions ν_A, ν_B respectively where

$\mu_A, \mu_B, \nu_A, \nu_B$: U $\to$ [0, 1].

$A \times B$ is the IFS in U x U defined by

$$\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$$

$$\nu_{A\times B}(x,y) = \max\{\nu_A(x), \nu_B(y)\}$$

for all x, y ∈ U.

Definition 2.1 (Varghese et al., 2012) Let $R \subseteq A \times B$

i.e., $\mu_R(x,y) \le \mu_{A\times B}(x,y)$ and

$\nu_R(x,y) \ge \nu_{A\times B}(x,y)$

with the condition that

$$0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$$

Then R is an IFR from A to B

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Definition 2.2 (Varghese et al., 2012)

Let R, R_1, R_2 be IFRs from A to B

Then $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 + R_2$, $R_1 . R_2$, $R_1 \bigcup R_2$, $R_1 \bigcap R_2$, R_1 □ R_2, $R_1 \otimes R_2$, $\overline{R}$ and R^{-1} are defined as follows :

1. $\mu_{R_1 \cup R_2}(x,y) = \max\left\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\right\}$
   $\nu_{R_1 \cup R_2}(x,y) = \min\left\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\right\}$

2. $\mu_{R_1 \cap R_2}(x,y) = \min\left\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\right\}$
   $\nu_{R_1 \cap R_2}(x,y) = \max\left\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\right\}$

3. $\mu_{R_1 + R_2}(x,y) = \mu_{R_1}(x,y) + \mu_{R_2}(x,y) - \mu_{R_1}(x,y)\mu_{R_2}(x,y)$
   $\nu_{R_1 + R_2}(x,y) = \nu_{R_1}(x,y)\nu_{R_2}(x,y)$

4. $\mu_{R_1 . R_2}(x,y) = \mu_{R_1}(x,y)\mu_{R_2}(x,y)$
   $\nu_{R_1 . R_2}(x,y) = \nu_{R_1}(x,y) + \nu_{R_2}(x,y) - \nu_{R_1}(x,y)\nu_{R_2}(x,y)$

5. $\mu_{R_1 \bigcup R_2}(x,y) = \min\left\{1, \mu_{R_1}(x,y) + \mu_{R_2}(x,y)\right\}$
   $\nu_{R_1 \bigcup R_2}(x,y) = \max\left\{0, \nu_{R_1}(x,y) + \nu_{R_2}(x,y) - 1\right\}$

6. $\mu_{R_1 \bigcap R_2}(x,y) = \max\left\{0, \mu_{R_1}(x,y) + \mu_{R_2}(x,y) - 1\right\}$
   $\nu_{R_1 \bigcap R_2}(x,y) = \min\left\{1, \nu_{R_1}(x,y) + \nu_{R_2}(x,y)\right\}$

7. $\mu_{R_1 \square R_2}(x,y) = \dfrac{\mu_{R_1}(x,y) + \mu_{R_2}(x,y)}{2}$
   $\nu_{R_1 \square R_2}(x,y) = \dfrac{\nu_{R_1}(x,y) + \nu_{R_2}(x,y)}{2}$

8. $\mu_{R_1 \otimes R_2}(x,y) = \sqrt{\mu_{R_1}(x,y)\mu_{R_2}(x,y)}$
   $\nu_{R_1 \otimes R_2}(x,y) = \sqrt{\nu_{R_1}(x,y)\nu_{R_2}(x,y)}$

9. $\mu_{\bar{R}}(x,y) = \min\left\{1 - \mu_R(x,y), \mu_{A\times B}(x,y)\right\}$
   $\nu_{\bar{R}}(x,y) = \max\left\{1 - \nu_R(x,y), \nu_{A\times B}(x,y)\right\} = C(x,y), if\, 0 \le \mu_{\bar{R}}(x,y) + C(x,y) \le 1$
   $\mu_R(x,y), \qquad if\, \mu_{\bar{R}}(x,y) + C(x,y) > 1$

10. $\mu_{R^{-1}}(x,y) = \mu_R(y,x)$
    $\nu_{R^{-1}}(x,y) = \nu_R(y,x)$

$$\forall x, y \in U$$

Note 1

If A and B are ordinary subsets of U, then
$\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\} = 1$
$\nu_{A\times B}(x,y) = \max\{\nu_A(x), \nu_B(y)\} = 0$

for all $x \in A$, $y \in B$. Then R is an IFR from A to B if $\mu_R(x,y) \le 1$, $\nu_R(x,y) \ge 0$, $0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$. This coincides with the definition of IFR in the cartesian product of universes.

Note 2

If A and B are fuzzy subsets of U, then this definition of R coincides with the definition of fuzzy relation in (Chakraborthy *et al*., 1983) where $\mu_R(x,y)$ d" $\mu_{A\times B}(x,y)$.

Note 3

If R is a relation from A to B, then R^{-1} is a relation from B to A

$\mu_{R^{-1}}(x,y) \le \mu_{B\times A}(x,y)$ as in (Chakraborthy *et al*., 1983)

$$\nu_{R^{-1}}(x,y) = \nu_R(y,x) \ge \max\left\{\nu_A(y), \nu_B(x)\right\}$$
$$= \nu_{B\times A}(x,y)$$

Note 4

We use the following matrix representation for membership and nonmembership functions. If the universal set U={a_1,a_2,……a_n} and if G is an IFS in U x U with membership function and nonmembership function , then and will be described in matrix notation as,

$$\mu_G : \begin{pmatrix} \mu_G(a_1,a_1) & \mu_G(a_2,a_1) & ....\mu_G(a_n,a_1) \\ \mu_G(a_1,a_2) & \mu_G(a_2,a_2) & ....\mu_G(a_n,a_2) \\ .... & .... & .... \\ \mu_G(a_1,a_n) & \mu_G(a_2,a_n) & ....\mu_G(a_n,a_n) \end{pmatrix}$$

and

$$\nu_G : \begin{pmatrix} \nu_G(a_1,a_1) & \nu_G(a_2,a_1) & ....\nu_G(a_n,a_1) \\ \nu_G(a_1,a_2) & \nu_G(a_2,a_2) & ....\nu_G(a_n,a_2) \\ .... & .... & .... \\ \nu_G(a_1,a_n) & \nu_G(a_2,a_n) & ....\nu_G(a_n,a_n) \end{pmatrix}$$

Theorem 2.1

If R_1 and R_2 are intuitionistic fuzzy relations from A to B, then

(i) $R_1 \subseteq R_2 \Rightarrow R_1^{-1} \subseteq R_2^{-1}$

(ii) $\left(R_1^{-1}\right)^{-1} = R_1$

(iii) $\left(R_1 * R_2\right)^{-1} = R_1^{-1} * R_2^{-1}$ where $*$ stands for $\cup, \cap, +, ., \bigcup, \bigcap, \square, \otimes$

Proof

For the case of membership function, the theorem has been proved (Chakraborthy *et al*., 1983) . But we prove the case of nonmembership function.

(i) $\mu_{R_1^{-1}}(y,x) \le \mu_{R_2^{-1}}(y,x)$

$\nu_{R_1^{-1}}(y,x) \ge \nu_{R_2^{-1}}(y,x)$ since $R_1 \subseteq R_2$

(ii) $\mu_{\left(R_1^{-1}\right)^{-1}}(x,y) = \mu_{R_1}(x,y)$

$\nu_{\left(R_1^{-1}\right)^{-1}}(x,y) = \nu_{R_1^{-1}}(y,x) = \nu_{R_1}(x,y)$

(iii) $\mu_{(R_1 * R_2)^{-1}}(x,y) = \mu_{R_1^{-1} * R_2^{-1}}(x,y)$

$\nu_{(R_1 * R_2)^{-1}}(x,y) = \nu_{R_1 * R_2}(y,x) = \nu_{R_1^{-1} * R_2^{-1}}(x,y)$

in all cases.

Definition 2.3 The composition of two IFRs R_1 and R_2 is defined by

$$\mu_{R_1 \circ R_2}(x,y) = \max_{z \in U}\left[\min\left(\mu_{R_1}(x,z), \mu_{R_2}(z,y)\right)\right] \quad \textit{and}$$

$$\nu_{R_1 \circ R_2}(x,y) = \min_{z \in U}\left[\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right)\right]$$

where R_1 is a relation from A to B and R_2 is a relation from B to C.

Lemma 2.1

If a, b, c, d, e, f, g, h $\in$ [0, 1], $0 \le$ a+e ≤ 1, $0 \le$ b+f ≤ 1, $0 \le$ c+g ≤ 1, $0 \le$ d+h ≤ 1, then max [min (a,b), min (c,d)] + min [max (e,f), max (g,h)] ≤ 1

Proof of the lemma :

We have, e $\le$ 1- a, f $\le$ 1- b

So max (e, f) $\le$ max (1-a, 1-b)

Similarly max (g, h) $\le$ max (1-c, 1-d)

min [max (e, f), max (g, h)] $\le$ min [max (1-a, 1-b), max (1-c, 1-d)]

So L.H.S. $\le$ max [min (a,b), min (c,d)] + min [max (1-a, 1-b), max (1-c, 1-d)]

= max [min (a,b), min (c,d)] + min [1 – min(a,b), 1 – min (c, d)]

= 1

This completes the proof of the lemma.

Theorem 2.2

Let R_1 be a relation from A to B and R_2 a relation from B to C,

then $R_1 \circ R_2$ is a relation from A to C.

Proof

$\mu_{R_1 \circ R_2}(x,y) \le \min\left[\mu_A(x), \mu_C(y)\right]$ as in (Chakraborthy *et al*., 1983)

For each $z \in U$,

$\nu_{R_1}(x,z) \ge \max\left[\nu_A(x), \nu_B(z)\right]$

$\nu_{R_2}(z,y) \ge \max\left[\nu_B(z), \nu_C(y)\right]$

In all the six cases of ordering of $\nu_A(x),\ \nu_B(z), \nu_C(y)$,

$$\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right) \ge \max\left[\nu_A(x), \nu_C(y)\right]$$

So $\min_{z \in U}\left[\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right)\right] \ge \max\left[\nu_A(x), \nu_C(y)\right]$

By lemma 2.1, it follows that

$$0 \le \mu_{R_1 \circ R_2}(x,y) + \nu_{R_1 \circ R_2}(x,y) \le 1$$

Some properties of composition are given in the next theorem.

Theorem 2.3

(i) R_1 o $R_2 \ne R_2$ o R_1 where R_1 and R_2 are IFRs on A.

(ii) (a) Let R_1 be a relation from A to B, R_2 and R_3 are relations from B to C and $*$ stands for any of $\cap, +, ., \bigcup, \bigcap, \square, \otimes$.

Then R_1 o ($R_2 * R_3$) $\ne$ (R_1 o R_2) $*$ (R_1 o R_3)

(b) R_1 o ($R_2 \cup R_3$) = (R_1 o R_2) $\cup$ (R_1 o R_3)

(iii) $R_1 \subseteq R_2 \Rightarrow R_1$ o $R_3 \subseteq R_2$ o R_3 where R_1, R_2 are relations from A to B and R_3 is a relation from B to C.

(iv) (R_1 o R_2)$^{-1}$ = R_2^{-1} o R_1^{-1} where R_1 is a relation from A to B and R_2 is relation from B to C

Proof

We will prove this by a counter example.

Let $U = \{a, b, c\}$ and A be given by

$\mu_A(a) = .7, \mu_A(b) = .8, \mu_A(c) = .8$

$\nu_A(a) = \nu_A(b) = .1, \nu_A(c) = .5$

Then

$$\mu_{A \times A} : \begin{pmatrix} .7 & .7 & .2 \\ .7 & .8 & .2 \\ .2 & .2 & .2 \end{pmatrix}, \nu_{A \times A} : \begin{pmatrix} .1 & .1 & .5 \\ .1 & .1 & .5 \\ .5 & .5 & .5 \end{pmatrix}$$

Let R_1, R_2 be relations on A defined by

$$\mu_{R_1}:\begin{pmatrix} .6 & .7 & .1 \\ .5 & .7 & .1 \\ .2 & .1 & .1 \end{pmatrix}, \mu_{R_2}:\begin{pmatrix} .6 & .6 & .1 \\ .5 & .8 & .14 \\ .2 & .1 & .1 \end{pmatrix}$$

$$\nu_{R_1}:\begin{pmatrix} .2 & .1 & .6 \\ .2 & .15 & .5 \\ .6 & .5 & .6 \end{pmatrix}, \nu_{R_2}:\begin{pmatrix} .15 & .1 & .55 \\ .2 & .1 & .5 \\ .6 & .6 & .5 \end{pmatrix}$$

It could be easily proved that $\circ$ is not commutative.

(ii) (a) can also be proved by another example.
Let $U=\{a,b,c\}$ and membership functions of A, B, C be given by

$$\mu_A(a)=\mu_A(b)=.96, \mu_A(c)=0$$
$$\mu_B(a)=\mu_B(b)=.95, \mu_B(c)=0$$
$$\mu_C(a)=\mu_C(b)=.92, \mu_C(c)=0$$

Then, $\mu_{A\times B}:\begin{pmatrix} .95 & .95 & 0 \\ .95 & .95 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu_{B\times C}:\begin{pmatrix} .92 & .92 & 0 \\ .92 & .92 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Let $R_1 : A \to B, R_2$ and $R_3 : B \to C$ defined by

$$\mu_{R_1}:\begin{pmatrix} .7 & .4 & 0 \\ .8 & .4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mu_{R_2}:\begin{pmatrix} .7 & .8 & 0 \\ .4 & .6 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mu_{R3}:\begin{pmatrix} .5 & .1 & 0 \\ .3 & .5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

With *R_1, R_2, R_3* as above one can check that $R_1 \circ (R_2 * R_3)$ is not distributive where $*$ is any of $\cap, +, ., \bigcup, \bigcap, \square$ *and* $\otimes$

(b) We can see that membership functions of $R_1 \circ (R_2 \cup R_3)$ *and* $(R_1 \circ R_2) \cup (R_1 \circ R_3)$ are equal as in (Chakraborthy *et al*., 1983)

The no**nmember**ship function of $R_1 \circ (R_2 \cup R_3)$ is

$$\min_{Z\in U}\left[\max\left(\nu_{R_1}(x,z), \nu_{R_2\cup R_3}(z,y)\right)\right]$$
$$=\min_{Z\in U}\left[\max\left\{\nu_{R_1}(x,z), \min\left(\nu_{R_2}(z,y), \nu_{R_3}(z,y)\right)\right\}\right]$$

For each z,

$$\max\left\{\nu_{R_1}(x,z), \min\left(\nu_{R_2}(z,y), \nu_{R_3}(z,y)\right)\right\}$$
$$=\min\left\{\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right), \max\left(\nu_{R_1}(x,z), \nu_{R_3}(z,y)\right)\right\}$$

$$\nu_{R_1\circ(R_2\cup R_3)}(x,y)$$
$$=\min_{Z\in U}\left[\min\left\{\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right), \max\left(\nu_{R_1}(x,z), \nu_{R_3}(z,y)\right)\right\}\right]$$

$$\nu_{(R_1\circ R_2)\cup(R_1\circ R_3)}(x,y)$$
$$=\min\left[\min_{Z\in U}\left\{\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right)\right\}, \min_{Z\in U}\left\{\max\left(\nu_{R_1}(x,z), \nu_{R_3}(z,y)\right)\right\}\right]$$
$$=\min_{Z\in U}\left[\min\left\{\max\left(\nu_{R_1}(x,z), \nu_{R_2}(z,y)\right), \max\left(\nu_{R_1}(x,z), \nu_{R_3}(z,y)\right)\right\}\right]$$
$$=\nu_{R_1\circ(R_2\cup R_3)}(x,y)$$

Hence (ii) is proved.

(iii) $\mu_{R_1\circ R_3}(x,y) \le \mu_{R_2\circ R_3}(x,y)$

as in (Chakraborthy *et al*., 1983)

$$\nu_{R_1\circ R_3}(x,y) = \min_{Z\in U}\left[\max\left(\nu_{R_1}(x,z), \nu_{R_3}(z,y)\right)\right]$$
$$\ge \min_{Z\in U}\left[\max\left(\nu_{R_2}(x,z), \nu_{R_3}(z,y)\right)\right]$$
$$=\nu_{R_2\circ R_3}(x,y)$$

$R_1 \circ R_3 \subseteq R_2 \circ R_3$

(iv)

$$\mu_{(R_1\circ R_2)^{-1}}(x,y) = \mu_{R_2^{-1}\circ R_1^{-1}}(x,y)$$

$$\nu_{(R_1\circ R_2)^{-1}}(x,y) = \nu_{R_1\circ R_2}(y,x)$$
$$=\min_{Z\in U}\left[\max\left(\nu_{R_1}(y,z), \nu_{R_2}(z,x)\right)\right]$$
$$=\min_{Z\in U}\left[\max\left(\nu_{R_2^{-1}}(x,z), \nu_{R_1^{-1}}(z,y)\right)\right]$$
$$=\nu_{R_2^{-1}\circ R_1^{-1}}(x,y)$$

This completes the proof of the theorem.

CONCLUSION

The IFRs presented in this paper are extensions of generalized fuzzy relations defined in (Chakraborthy *et al*.,1983)and these are generalizations of IFRs already in existence.

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